Dynamic Programming (DP)

Background

- Recursive methods
  - Similar subproblems inside the original problem
  - Not always good
    - Too many redundant calls

Cons and Pros of Recursion

- Recursion is good for these:
  - Sort algorithms such as Quick Sort, Merge Sort, etc.
  - Factorial (Well, actually NO in pratcice!)
  - DFS over graphs
  - …
  - (In REAL programming practice, recursion is mostly avoided!)
- But bad for these:
  - Fibonacci numbers
  - Optimal order of matrices multiplication
  - …

Introduction: Fibonacci

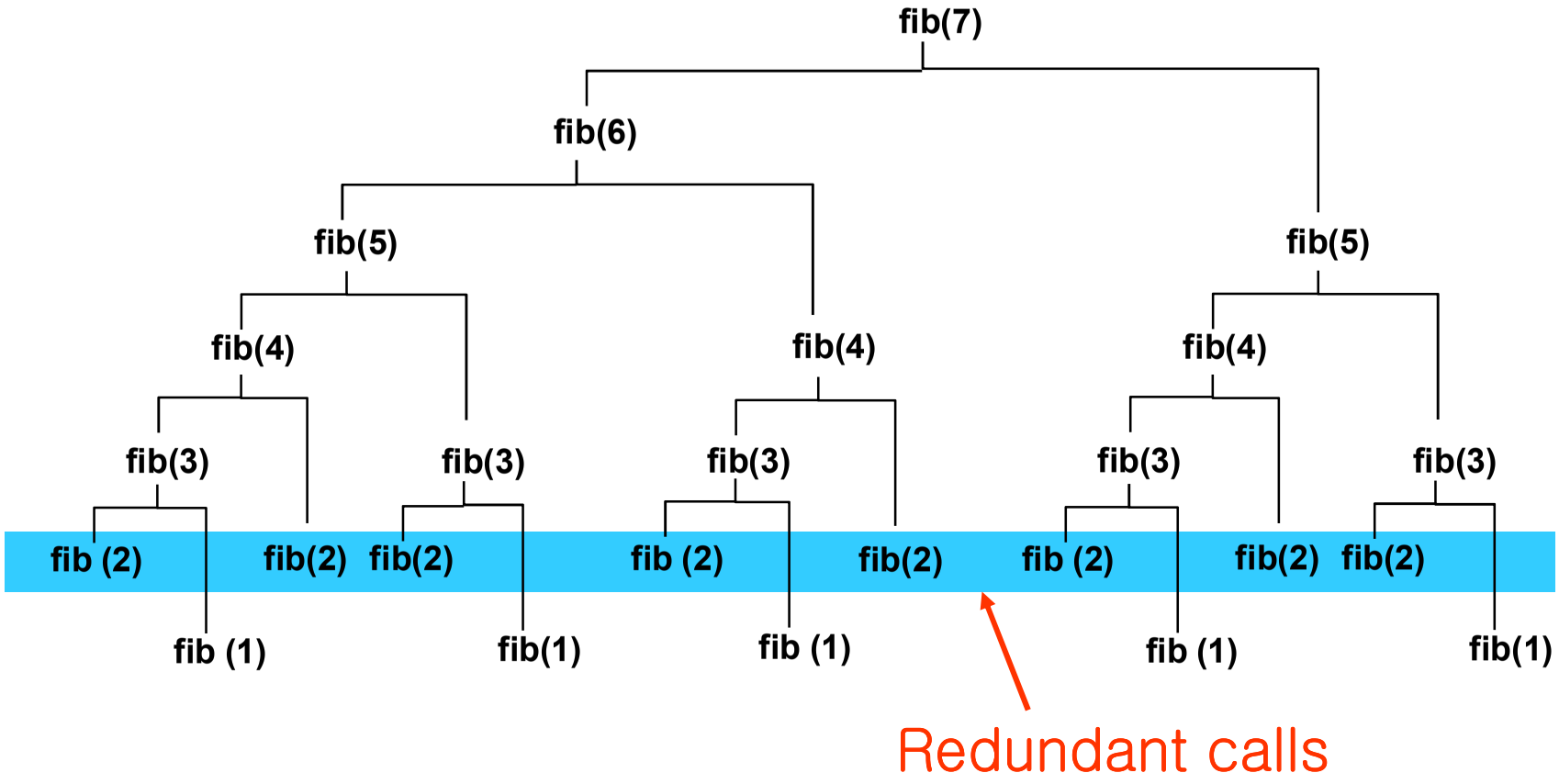
- $f_n = f_{n-1} + f_{n-2}$
- Very simple problem
  - But contains good motivations and implementation details of Dynamic programming
- Fibonacci is a fast growing function!

Recursive Algorithm for Fibonacci

```
fib(n)
{
        if (n = 1 or n = 2)
                then return 1;
                 else return (fib(n-1) +fib(n-2));
}
```

✓ Many redundant recursive calls

Call Tree of Fibonacci



DP Algorithm for Fibonacci

```
fibonacci(n)
{
        f[1] ← f[2] ← 1;
        for i ← 3 to n
                f[i] ← f[i-1] +f[i-2];
        return f[n];
}
```

✓ $\Theta(n)$ time complexity

We need DP when…

- Optimal substructure
  - Optimal solution has optimal sub-solution
- Overlapping recursive calls
  - In recursive solution, too many redundant recursive calls for the same subproblem

➡ DP is a solution!

DP's property

- DP is implicitly related with DAG data structure
  - page 170 of DPV
- Comparison with Recursion
  - First, it gets its intermediate values using table lookup instead of recursive calls.
  - Second, it updates the parent field of each step, which will enable us to reconstruct the solution later.
  - Third, it is instrumented using a more general function instead of just returning the intermediate value. This will enable us to apply the routine to a wider class of problems.

Example 1:Pebbling a checkerboard

- In each cell of a 3×N table, there is an integer number, either positive or negative
- You put a pebble on a cell under these condition:
  - At each column, there should be at least one pebble
  - There should be no pebbles horizontally or vertically adjacent to a pebble
- Goal: Place pebbles so that the sum of numbers in the cells filled with pebbles is maximum

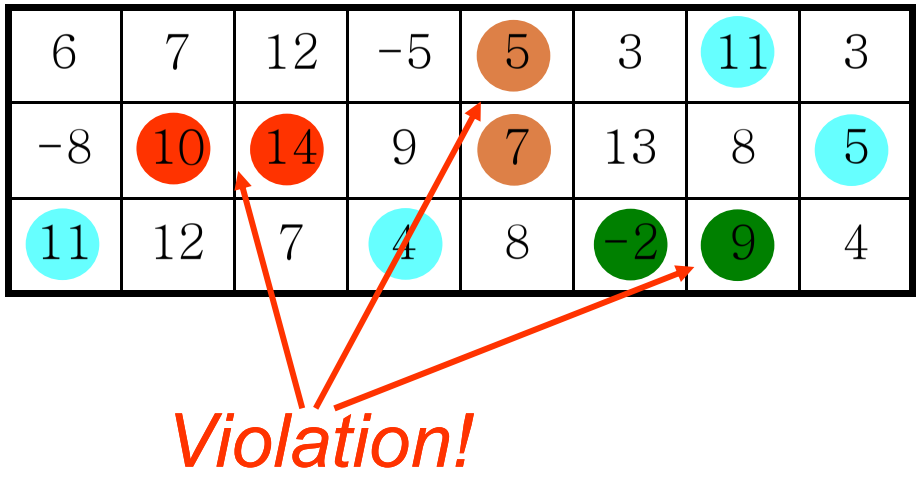
Example

6	7	12	-5	5	3	11	3
-8	10	14	9	7	13	8	5
11	12	7	4	8	-2	9	4

Legal

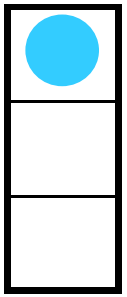
6	7	12	-5	5	3	11	3
-8	10	14	9	7	13	8	5
11	12	7	4	8	-2	9	4

Illegal



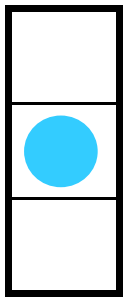
Possible patterns

Pattern 1:



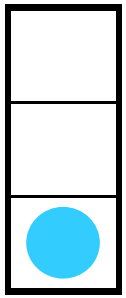
6	7	12	**-5**	5	3	11	3
-8	10	14	9	7	13	8	5
11	12	7	4	8	-2	9	4

Pattern 2:



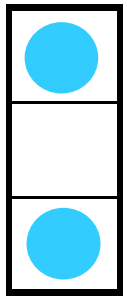
6	7	12	-5	5	3	11	3
-8	10	14	**9**	7	13	8	5
11	12	7	4	8	-2	9	4

Pattern 3:



6	7	12	-5	5	3	11	3
-8	10	14	9	7	13	8	5
11	12	7	**4**	8	-2	9	4

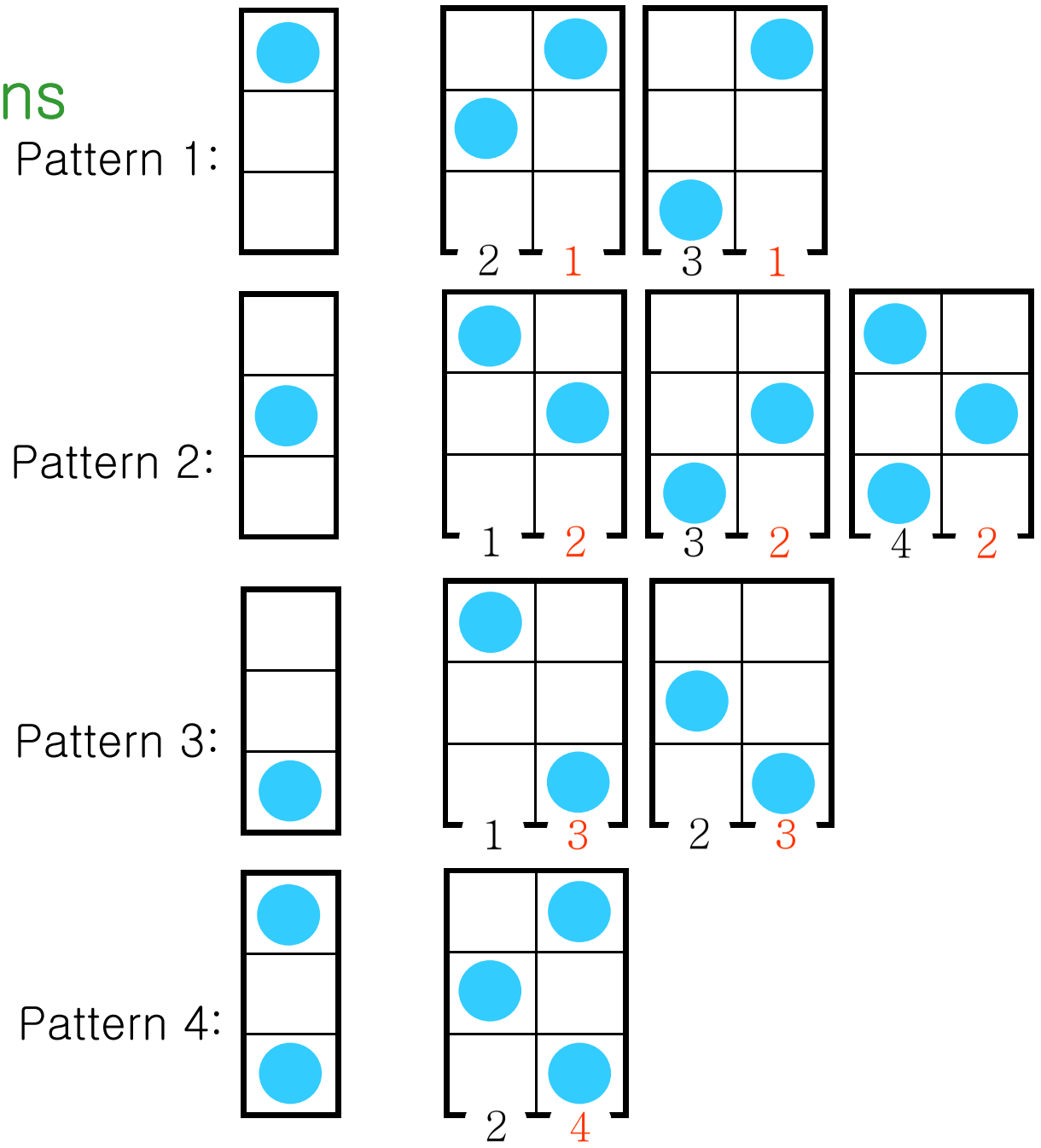
Pattern 4:



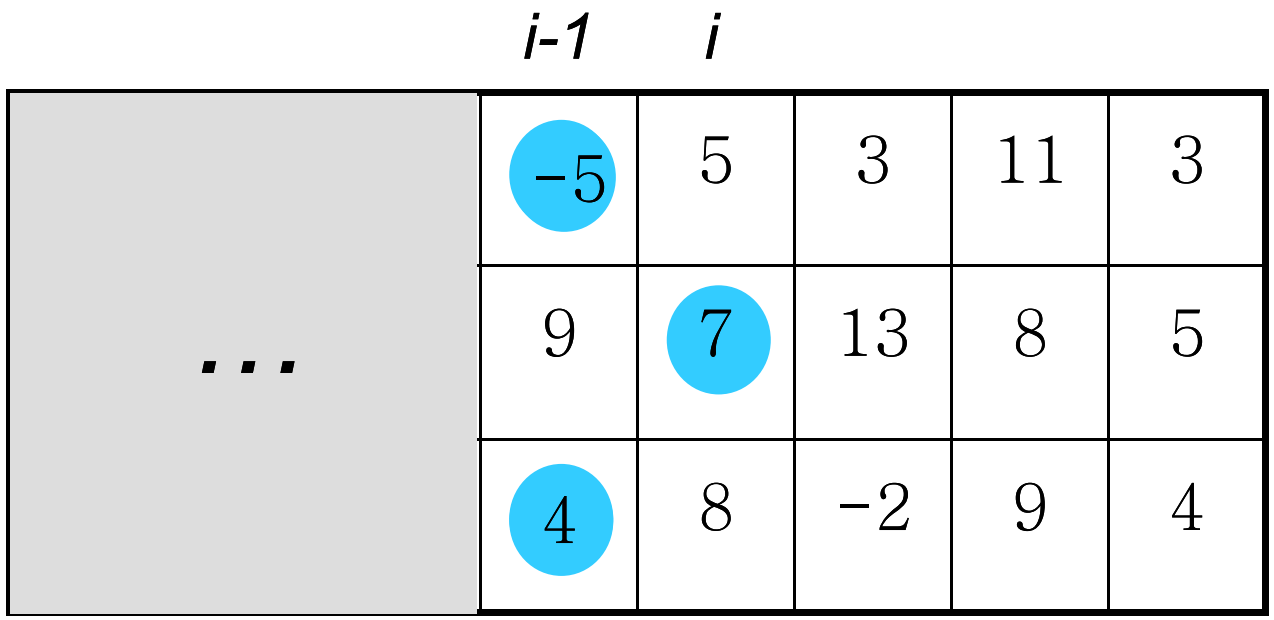
6	7	12	**-5**	5	3	11	3
-8	10	14	9	7	13	8	5
11	12	7	**4**	8	-2	9	4

There are only four patterns to fill one column.

Compatible Patterns



Pattern 1 is compatible with 2, and 3
Pattern 2 is compatible with 1, 3, and 4
Pattern 3 is compatible with 1, and 2
Pattern 4 is compatible with 2



Consider the relation between column i and column i-1.

Recursive Version

```
pebble(i, p)
▷ Calculate the max sum upto the column i when pattern p is on column i.
▷ w[i, p] : sum of pebbled integers when pattern p is on column i: p ∈ {1, 2, 3, 4}
{
    if (i = 1)
        then return w[1, p] ;
        else {
            max ← −∞ ;
            for q ← 1 to 4 {
                if (pattern q is compatible with pattern p)
                then {
                    tmp ← pebble(i−1, q) ;
                    if (tmp > max) then max ← tmp ;
                }
            }
            return (w[i, p] + max) ;
        }
}
```

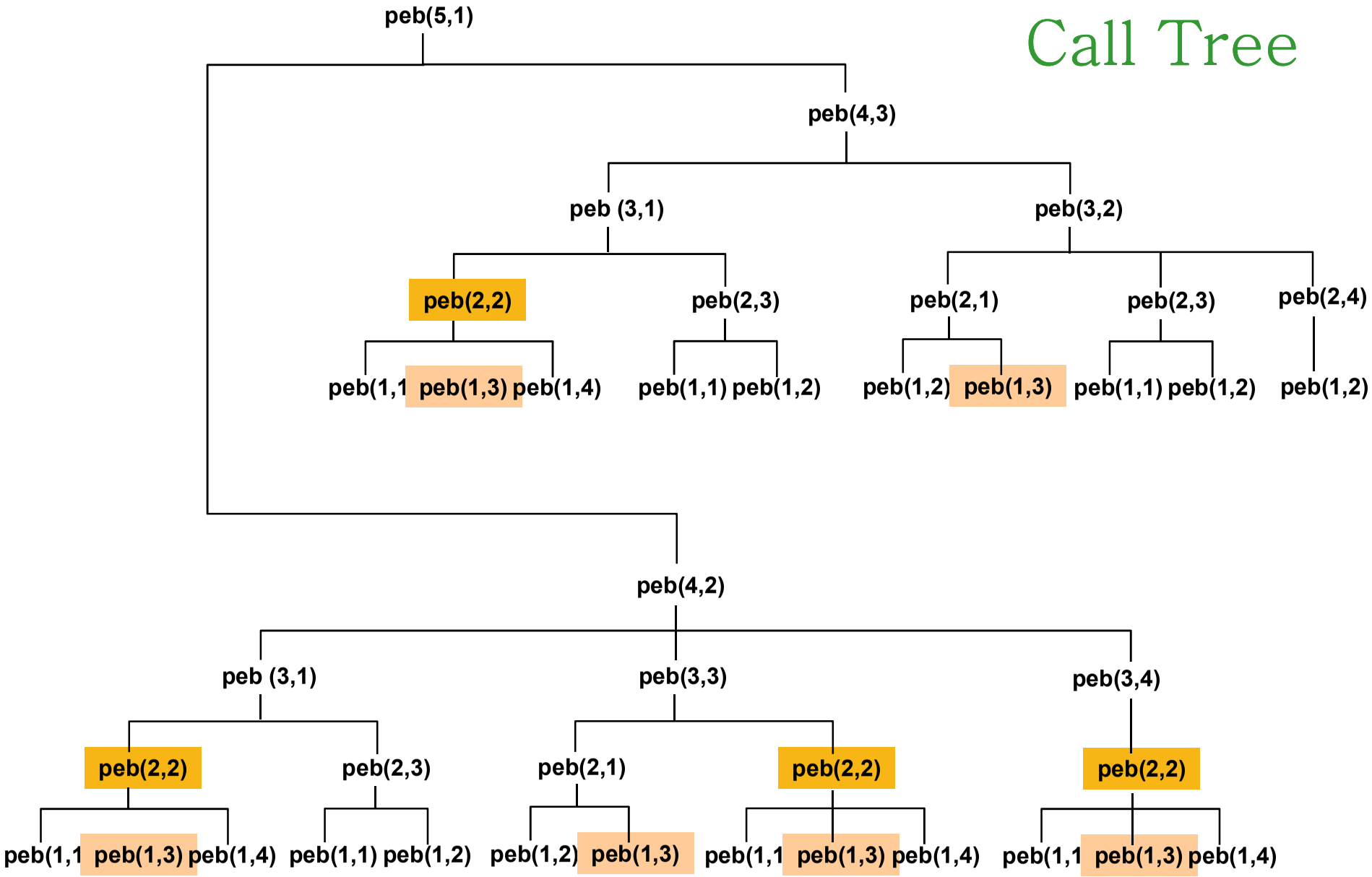
pebbleSum(*n*)

▷ Calculate the max sum upto column *n*

```
{
        return  max { pebble(n, p) } ;
                p =1,2,3,4
}
```

✓ The final answer is the max among pebble(*i*, *1*), …, pebble(*i*, *4*)

Call Tree


peb(5,1)
peb(4,3)
peb (3,1)
peb(3,2)
peb(2,2)
peb(2,3)
peb(2,1)
peb(2,3)
peb(2,4)
peb(1,1
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peb(1,1
peb(1,3)
peb(1,4)

Applying DP

- DP's properties
  - Optimal substructure
    - pebble(*i*, .) has pebble(*i*-1, .)
    - So, big problem's optimal solution contains smaller problem's optimal solutions
  - Overlapping recursive calls
    - Recursive version has too many redundant calls

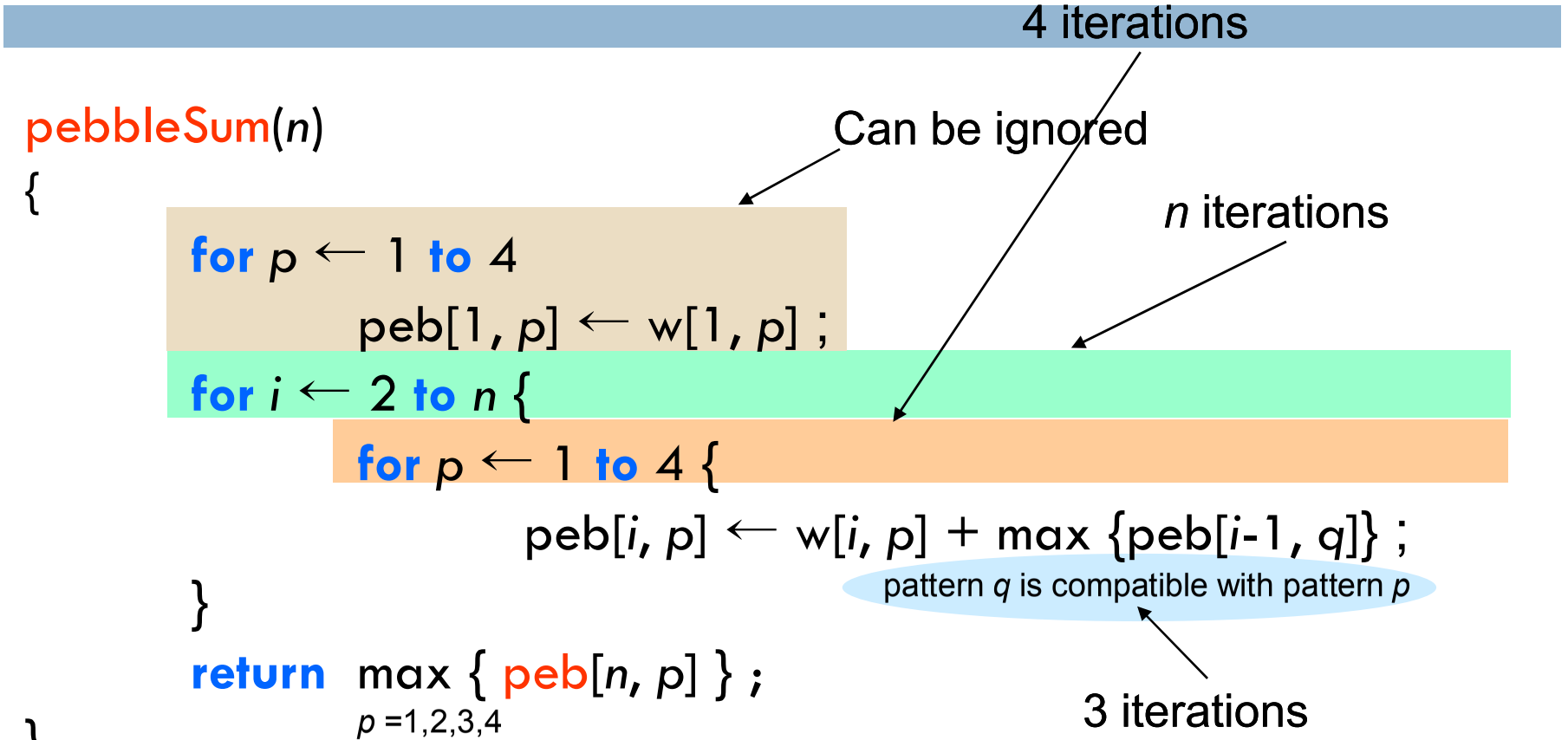
DP

```
pebbleSum(n)
{
        for p ← 1 to 4
                peb[1, p] ← w[1, p] ;
        for i ← 2 to n {
                for p ← 1 to 4 {
                        peb[i, p] ← w[i, p] + max {peb[i-1, q]} ;
        }
        return  max { peb[n, p] } ;
                p =1,2,3,4
}
```

For each pattern q which is compatible with pattern p

✓Complexity : $O(n)$

Complexity

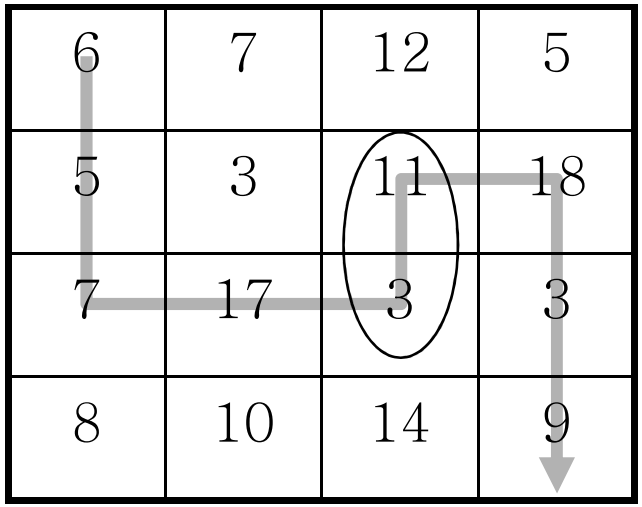


✓Complexity : $O(n)$ n * 4 * 3 = $O(n)$

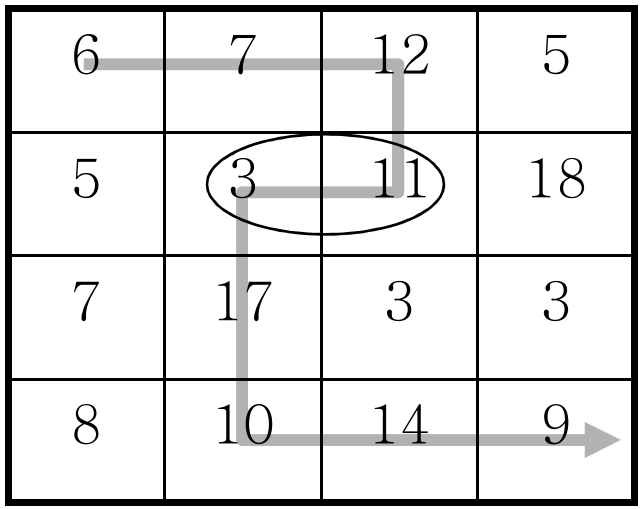
Example 2: Path in the array

- Given $n \times n$ matrix whose elements are integers, the algorithm moves from the top-left corner to the bottom-right corner of the matrix.
- Move with the following constraints:
  - Move to the right direction or the downward direction
  - Cannot move to the left, upward, or diagonal directions
- Goal: Move from the top-left corner to the bottom-right corner so that the sum of the numbers along the visited cells is the maximum

Illegal moves



Going up



Going left

Valid moves

6	7	12	5
5	3	11	18
7	17	3	3
8	10	14	9

6	7	12	5
5	3	11	18
7	17	3	3
8	10	14	9

Recursive Algorithm

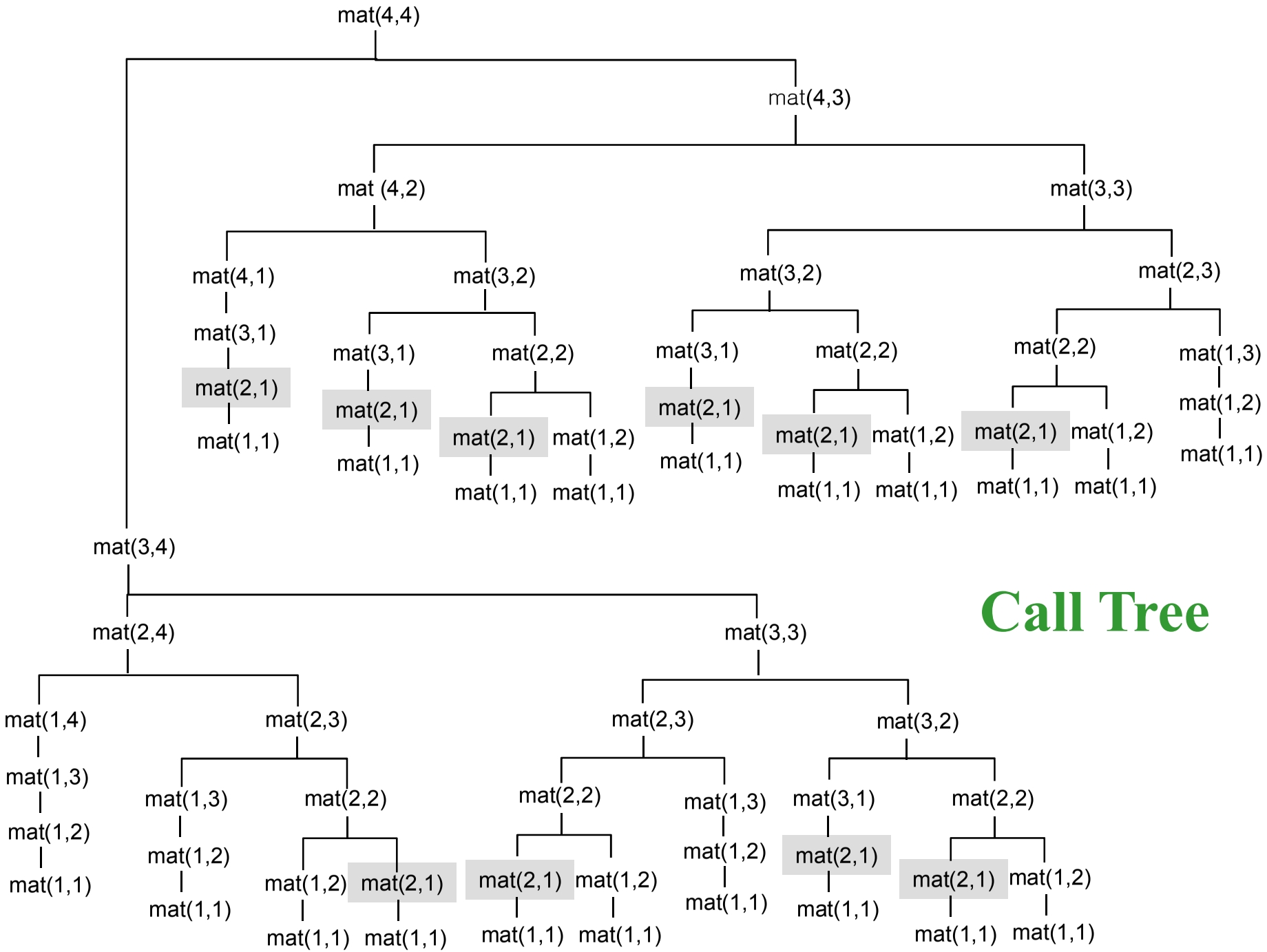
matrixPath(i, j)

▷ The max sum at the cell (i, j)

{

```
if (i = 1 and j = 1) then return m_ij;
else if (i = 1) then return (matrixPath(1, j-1) + m_ij);
else if (j = 1) then return (matrixPath(i-1, 1) + m_ij);
else return ((max(matrixPath(i-1, j), matrixPath(i, j-1)) + m_ij);
```

}


mat(4,4)
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mat(1,1)
mat(1,1)
Call Tree

DP

matrixPath(i, j)

▷ The max sum at the cell (i, j)

{

c[1,1] ← m_{11} ;

for i ← 2 **to** n

c[i,1] ← m_{i1} + c[i-1,1];

for j ← 2 **to** n

c[1, j] ← m_{1j} + c[1, j-1];

for i ← 2 **to** n

for j ← 2 **to** n

c[i, j] ← m_{ij} + max(c[i-1, j], c[i, j-1]);

return c[n, n];

}

Example 3: Matrix-Chain Multiplication

- Matrices A, B, C
  - (AB)C = A(BC)
- Ex: A:10 × 100, B:100 × 5, C:5 × 50
  - (AB)C: 7500 times multiplications
  - A(BC): 75000 times multiplications
- What is the best order to multiply A_1, A_2, A_3, …, A_n?

Recursive Relation

- At the time of the last matrix multiplication
  - n-1 possibilities
    - $(A_1 \ldots A_{n-1})A_n$
    - $(A_1 \ldots A_{n-2})\ (A_{n-1}A_n)$
    - . . .
    - $(A_1A_2)(A_3 \ldots A_n)$
    - $A_1(A_2 \ldots A_n)$
  - Which one is most interesting?

- m[i, j]: The cost of multiplying $A_i, A_{i+1}, \ldots, A_j$
- A_k's dimension: $p_{k-1}p_k$

$$m[i,j] = \begin{cases} 0 & , i=j \\ \min\limits_{i \le k \le j-1} \{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & , i<j \end{cases}$$

Recursive Algorithm

```
rMatrixChain(i, j)
▷ Calculate the minimum cost to multiply matrices
{
    if (i = j) then return 0;    ▷ 0 when there is only one matrix
    min ← ∞;
    for k ← i to j-1 {
        q ← rMatrixChain(i, k) + rMatrixChain(k+1, j) + p_{i-1} p_k p_j;
        if (q < min) then min ← q;
    }
    return min;
}
```

✓ Too many redundant recursive calls!

DP

```
matrixChain(i, j)
{
        for i ← 1 to n
                m[i, i] ← 0;            ▷ 0 when there is only one matrix
        for r ← 1 to n-1    ▷ Size of the problem is r+1
                for i ← 1 to n-r {
                        j ← i+r;
                        m[i, j] ← min {m[i, k] + m[k+1, j] + p_{i-1} p_k p_j};
                                  i ≤ k ≤ j-1
                }
        return m[1, n];
}
```

✓ Complexity: $\Theta(n^3)$

Example 4:
Longest Common Subsequence(LCS)

- Find the longest common subsequence between two strings
- Example of subsequence
  - <bcdb> is a subsequence of <abcbdab>
- Example of Common subsequence
  - <bca> is a common subsequence of strings <abcbdab> and <bdcaba>
- Longest common subsequence(LCS)
  - The longest among common subsequences
  - Ex: <bcba> is an LCS of string <abcbdab> and <bdcaba>

Optimal Substructure

- Two strings $X_m = \langle x_1 x_2 \ldots x_m \rangle$ and $Y_n = \langle y_1 y_2 \ldots y_n \rangle$
  - if $x_m = y_n$, the length of LCS of X_m and Y_n is 1+ LCS of X_{m-1} and Y_{n-1}
  - if $x_m \neq y_n$, the length of LCS of X_m and Y_n is max of { LCS of X_m and Y_{n-1}, LCS of X_{m-1} and Y_n }

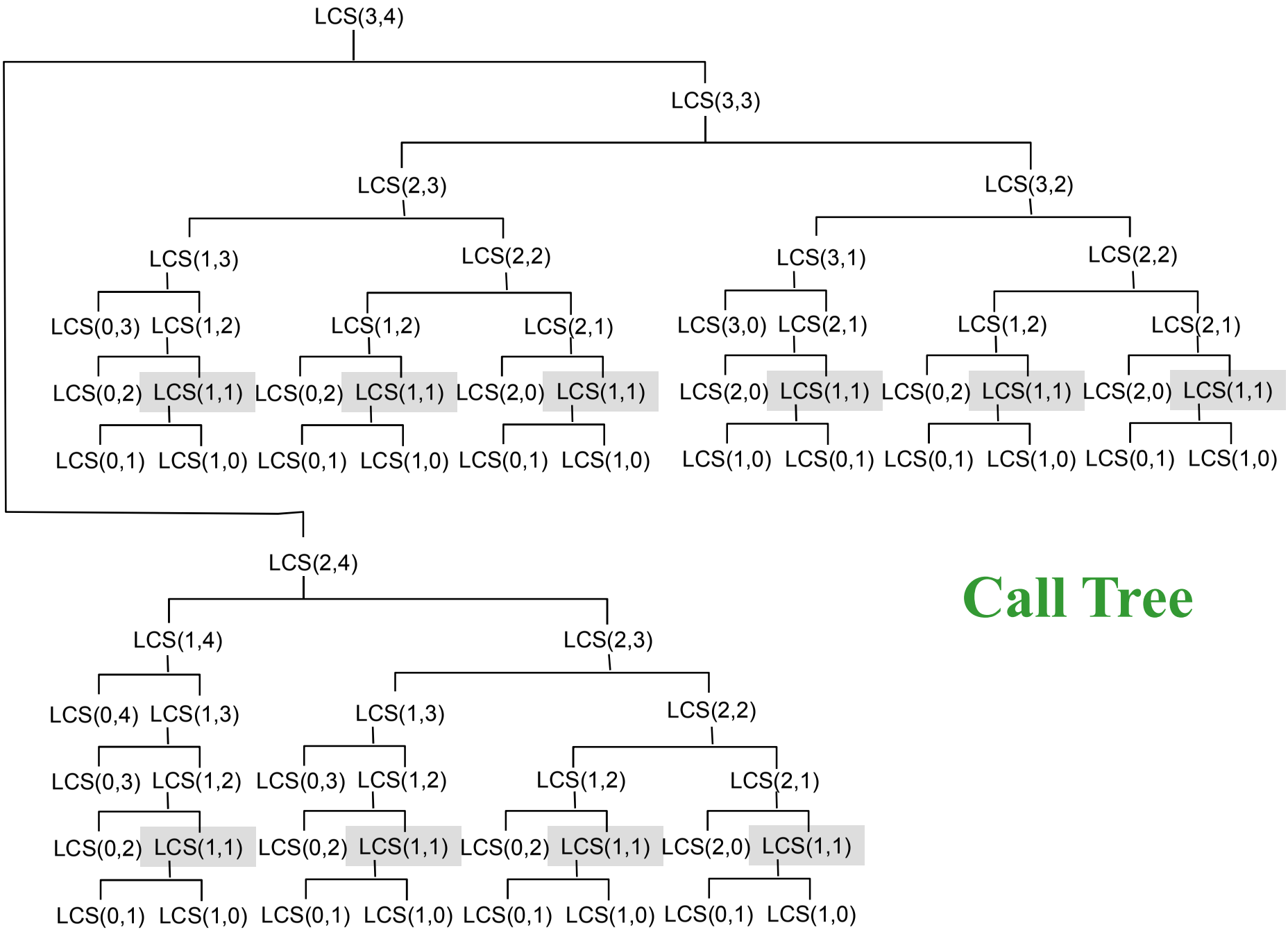
- $$c_{ij} = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c_{i-1,\,j-1} + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{c_{i-1,\,j},\ c_{i,\,j-1}\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

✓ c_{ij} : The length of LCS of two strings $X_i = \langle x_1 x_2 \cdots x_i \rangle$ and $Y_j = \langle y_1 y_2 \cdots y_j \rangle$

Recursive Algorithm

```
LCS(m, n)
▷ Calculate the length of LCS of two strings X_m and Y_n
{
    if (m = 0 or n = 0) then return 0;
    else if (x_m = y_n) then return LCS(m-1, n-1) + 1;
    else return max(LCS(m-1, n), LCS(m, n-1));
}
```

✓ Many redundant recursive calls!


LCS(3,4)
LCS(3,3)
LCS(2,3)
LCS(3,2)
LCS(1,3)
LCS(2,2)
LCS(3,1)
LCS(2,2)
LCS(0,3) LCS(1,2)
LCS(1,2) LCS(2,1)
LCS(3,0) LCS(2,1)
LCS(1,2) LCS(2,1)
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LCS(0,1) LCS(1,0)
LCS(0,1) LCS(1,0)

Call Tree

DP

```
LCS(m, n)
▷ Calculate the length of LCS of two strings X_m and Y_n
{
    for i ← 0 to m
        C[i, 0] ← 0;
    for j ← 0 to n
        C[0, j] ← 0;
    for i ← 1 to m
        for j ← 1 to n
            if (x_m = y_n) then C[i, j] ← C[i-1, j-1] + 1;
            else C[i, j] ← max(C[i-1, j], C[i, j-1]);
    return C[m, n];
}
```

✓ Complexity: $\Theta(mn)$

Maximum Monotone Subsequence

- A numerical sequence is monotonically increasing if the i^{th} element is at least as big as the $(i-1)^{st}$ element.
- The maximum monotone subsequence problem seeks to delete the fewest number of elements from an input string S to leave a monotonically increasing subsequence.
- Thus a longest increasing subsequence of "243517698" is "23568."

Reduction of Maximum Monotone Subsequence to LCS

- In fact, this is just a longest common subsequence problem, where the second string is the elements of S sorted in increasing order.
- Any common sequence of these two must
  - (a) represent characters in proper order in S, and
  - (b) use only characters with increasing position in the collating sequence, so the longest one does the job.

Example 5: Shortest Path

- Weighted digraph $G=(V, E)$
  - $w_{i,j}$: The edge length from vertex i to vertex j
    - No Edge : ∞
- Goal
  - Calculate all the shortest path lengths from the starting vertex *s* to the other vertices

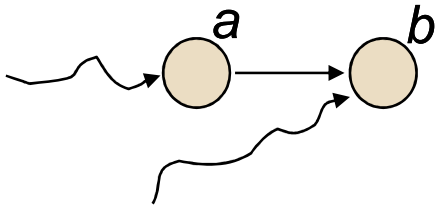
- d_t^k : Shortest path length from vertex s to vertex t through at most k edges
- Goal: d_t^{n-1}
- For $i \neq s$,
  - $d_t^0 = \infty$
  - $d_t^1 = w_{s,t}$

Recursive Relation

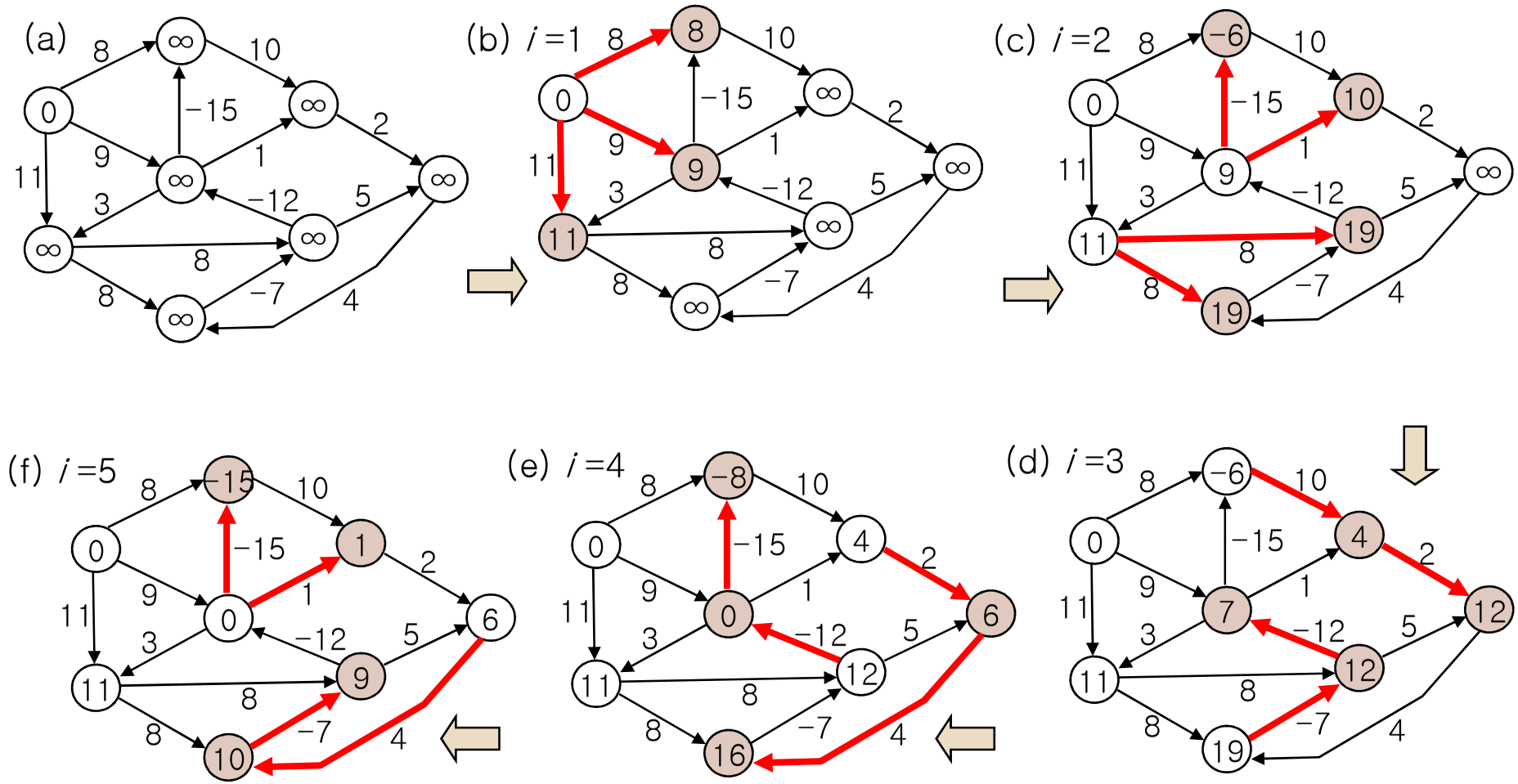
$$\begin{cases} d_t^{\,k} = \min_{\text{for all edges } (r,\,t)} \{d_r^{\,k-1} + w_{r,\,t}\} \\ d_s^{\,0} = 0; \\ d_t^{\,0} = \infty; \end{cases}$$

DP

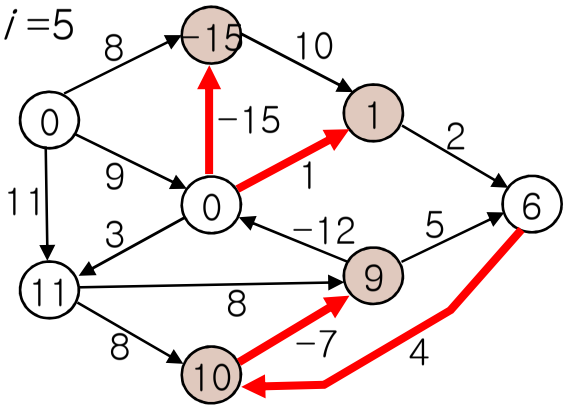
```
Bellman-Ford(G, s)
{
        d_s ← 0;
        for all vertices i ≠ s
                d_i ← ∞;
        for k ← 1 to n-1 {
                for all edges (a, b) {
                        if (d_a + w_a,b < d_b )  then d_b ← d_a + w_a,b ;
                }
        }
}
```



✓ Think of “propagation”!

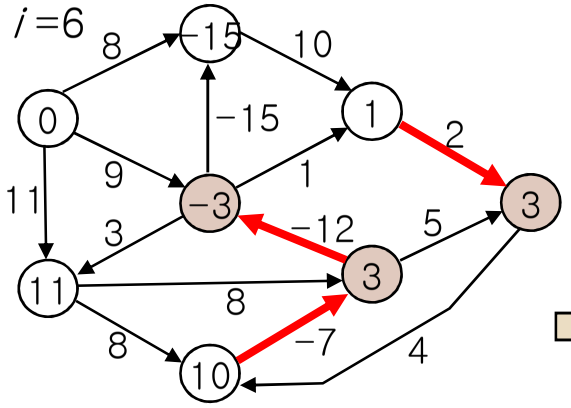

(a)
(b) i =1
(c) i =2
(d) i =3
(e) i =4
(f) i =5

(f) $i=5$

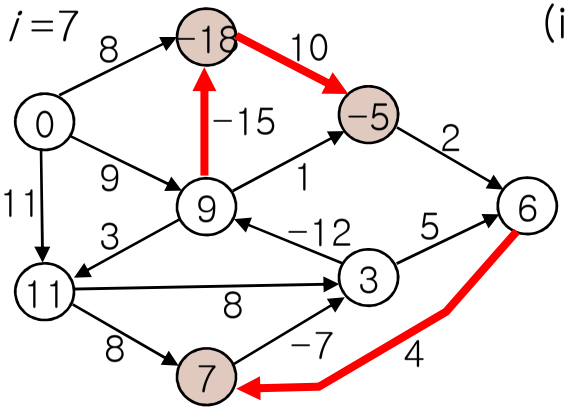


We discussed this
in the graph section

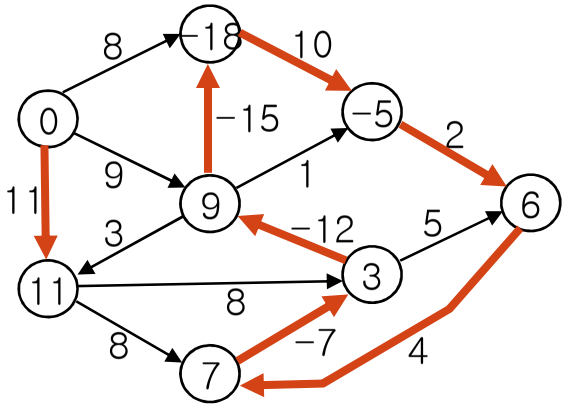
(g) $i=6$



(h) $i=7$



(i)



Example: Interwoven Strings

- Suppose you are given three strings of characters: X, Y , and Z, where $|X| = n$, $|Y| = m$, and $|Z| = n + m$. Z is said to be a shuffle of X and Y iff Z can be formed by interleaving the characters from X and Y in a way that maintains the left-to-right ordering of the characters from each string.

1. Show that cchocohilaptes is a shuffle of chocolate and chips, but chocochilatspe is not.
2. The strings abac and bbc occur interwoven in cabbabccdw.
3. Give an efficient dynamic-programming algorithm that determines whether Z is a shuffle of X and Y . (Hint: The values of the dynamic programming matrix you construct should be Boolean, not numeric.)